Use the radioactive decay model of exercise 7 and the rate of decay for C_{14} that you can compute there to determine the "time since death" when the following percentages of C_{14} remain in a given material.

(a) 88% (b) 12% (c) 2% (d) 98%

Solution:

The radioactive decay model is the first-order initial-value problem

$$\frac{dr}{dt} = -\lambda r \quad \text{and} \quad r(0) = r_0$$

whose solution is

$$r(t) = r_0 e^{-\lambda t}$$

where $r(t)$ represents the amount of the radioactive isotope present at time t and the parameter $-\lambda$ (with $\lambda > 0$) represents the decay rate (from exercise 6).

To determine the value of λ for C_{14}, we use that the half-life of C_{14} is $5,230$ years, so $r(5,230) = 0.5r_0$ and

$$0.5r_0 = r_0 e^{-\lambda(5,230)} \quad \longrightarrow \quad 0.5 = e^{-\lambda(5,230)} \quad \longrightarrow \quad \ln(0.5) = -\lambda(5,230) \quad \longrightarrow \quad \lambda = -\frac{\ln(0.5)}{5,230} \simeq 0.0001325329\ldots$$

Now we can solve for t when $r(t) = 0.88r_0$, etc. These values can also be read from an accurate graph of $r(t) = r_0 e^{-\lambda t}$.

$$0.88r_0 = r_0 e^{-\left(-\frac{\ln(0.5)}{5,230}\right)t} \quad \longrightarrow \quad 0.88 = e^{\left(\frac{\ln(0.5)}{5,230}\right)t} \quad \longrightarrow \quad \ln(0.88) = \left(\frac{\ln(0.5)}{5,230}\right)t \quad \longrightarrow \quad t = \frac{5,230\ln(0.88)}{\ln(0.5)} \simeq 965 \text{ years}$$

$$0.12r_0 = r_0 e^{-\left(-\frac{\ln(0.5)}{5,230}\right)t} \quad \longrightarrow \quad 0.12 = e^{\left(\frac{\ln(0.5)}{5,230}\right)t} \quad \longrightarrow \quad \ln(0.12) = \left(\frac{\ln(0.5)}{5,230}\right)t \quad \longrightarrow \quad t = \frac{5,230\ln(0.12)}{\ln(0.5)} \simeq 15,998 \text{ years}$$

$$0.02r_0 = r_0 e^{-\left(-\frac{\ln(0.5)}{5,230}\right)t} \quad \longrightarrow \quad 0.02 = e^{\left(\frac{\ln(0.5)}{5,230}\right)t} \quad \longrightarrow \quad \ln(0.02) = \left(\frac{\ln(0.5)}{5,230}\right)t \quad \longrightarrow \quad t = \frac{5,230\ln(0.02)}{\ln(0.5)} \simeq 29,517 \text{ years}$$

$$0.98r_0 = r_0 e^{-\left(-\frac{\ln(0.5)}{5,230}\right)t} \quad \longrightarrow \quad 0.98 = e^{\left(\frac{\ln(0.5)}{5,230}\right)t} \quad \longrightarrow \quad \ln(0.98) = \left(\frac{\ln(0.5)}{5,230}\right)t \quad \longrightarrow \quad t = \frac{5,230\ln(0.98)}{\ln(0.5)} \simeq 152 \text{ years}$$