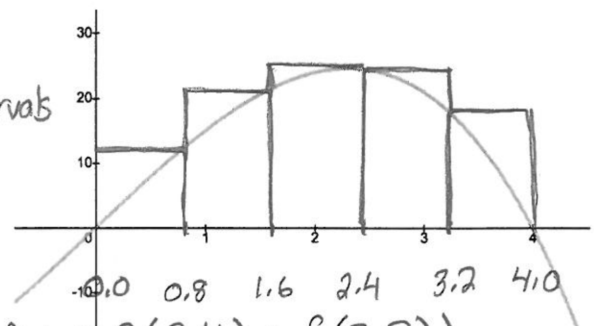
MTH 161 Section WN110
Exam 1
Winter 2020

Name: REDMAN
ID: KEY

Instructions: Show all of your work on the blank paper provided. Label your problems clearly and indicate each answer clearly.

1. For the function $f(x) = 16x - x^3$, make a reasonable estimate of the area under the function and above the x-axis over the interval $[0, 4]$ using five rectangles of equal width based on the x-axis that completely contain the area. *(6 points)*

   To help you visualize the area a graph is provided on the right.



dividing $[0,4]$ into 5 equal sub-intervals
$a = 0.0 < 0.8 < 1.6 < 2.4 < 3.2 < 4.0$

five rectangles of area

$$\left.\begin{array}{l} 0.8 * f(0.8) \\ 0.8 * f(1.6) \\ 0.8 * f(2.4) \\ 0.8 * f(2.4) \\ 0.8 * f(3.2) \end{array}\right\} \quad 0.8(f(0.8) + f(1.6) + 2f(2.4) + f(3.2)) \cong \boxed{81.1 \text{ square units}}$$

similar estimates are acceptable

maybe we could enlarge the third rectangle, but our estimate is still fair.

2. Fill in the following table of values, rounding to eight digits, and use that information to estimate the value of the limit $\lim_{x \to 2} \frac{x^2 - 4}{x^2 + x - 6}$. *(10 points)*

Notice $\frac{(x-2)(x+2)}{(x+3)(x-2)} = \frac{x+2}{x+3}$

x	$\frac{x^2-4}{x^2+x-6}$	x	$\frac{x^2-4}{x^2+x-6}$
1.9	0.7959 1837	2.1	0.8039 2157
1.99	0.7995 9920	2.01 2.11	0.8003 9920 0.8043 0528
1.999	0.7999 5999	2.001 2.111	0.8000 3999 0.8043 4357
1.9999	0.7999 9600	2.0001 2.1111	0.8000 0400 0.8043 4740

estimate:

$$\boxed{\lim_{x \to 2} \frac{x^2-4}{x^2+x-6} \cong 0.8}$$

$\left(\frac{4}{5}\right)$

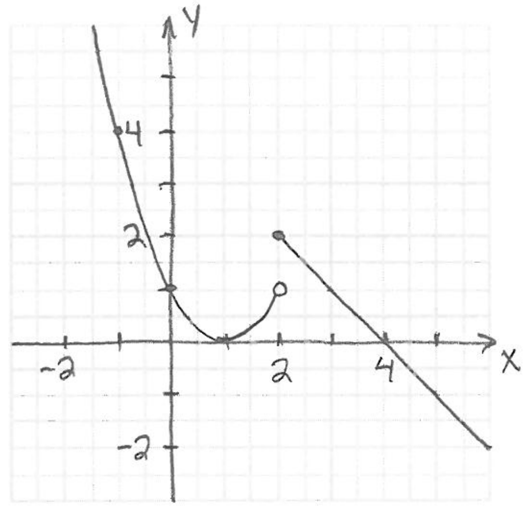
(our intention was to get closer to 2 from left and right)

2.1, 2.01, 2.001, 2.0001 would have been correct.

3. Make an accurate sketch of the following function and evaluate the limits below. *(12 points)*

$$f(x) = \begin{cases} x^2 - 2x + 1 & \text{if} \quad x < 2 \\ 4 - x & \text{if} \quad x \geq 2 \end{cases}$$

Notice: $x^2 - 2x + 1 = (x-1)^2$



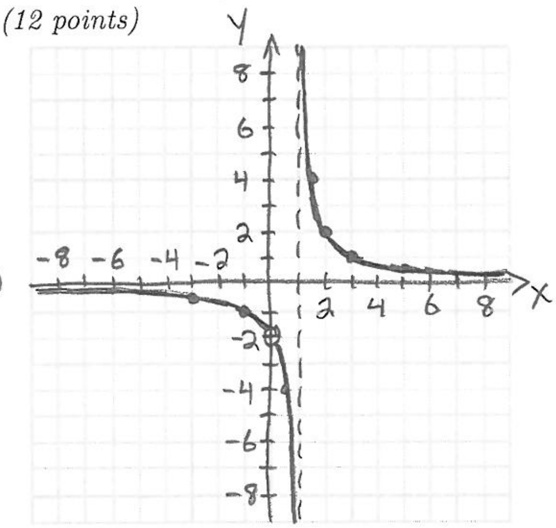
(a) $\lim_{x \to 2^-} f(x)$ $\boxed{= 1}$

(b) $\lim_{x \to 2^+} f(x)$ $\boxed{= 2}$

(c) $\lim_{x \to 2} f(x)$ $\boxed{\text{does not exist}}$

4. Make an accurate sketch of the following function. State the values of x (if any) at which the function is discontinuous, briefly state the reason it is discontinuous, and the nature of the disconitinuity, jump, removable, infinite, or other. *(12 points)*

$$f(x) = \frac{2x}{x^2 - x}$$



Notice $\frac{2x}{x(x-1)} = \frac{2}{x-1}$ if $x \neq 0$

= undefined if $x = 0$

$f(x)$ is also undefined at $x = 1$

x	reason	nature
0	$f(0)$ does not exist or $f(x)$ is not defined at $x=0$	removable
1	$\lim_{x \to 1} f(x)$ does not exist	infinite

5. Prove that $\lim_{x \to 2} x^2 - 3x - 1 = -3$ using the ϵ-δ definition of limit. *(6 points)*

Definition - Limit

$\lim_{x \to a} f(x) = L$ if and only if

for every $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |x - a| < \delta$ ensures $|f(x) - L| < \epsilon$.

our problem:

$f(x) = x^2 - 3x - 1$

$L = -3$

$a = 2$

$|x-a|$ is $|x-2|$

$|f(x)-L|$ is $|x^2-3x+2|$

is $|x-2||x-1|$

Proof let $\varepsilon > 0$, choose $\delta = \min\{1, \varepsilon/2\}$

$|f(x)-L| = |x^2-3x-2| = |x-2||x-1|$ we can control this because

$0 < |x-2| < \delta \leq 1$ forces x to be within 1 of 2: $1 < x < 3$

which in turn forces $0 < x-1 < 2$ so $|x-1| < 2$

$0 < |x-2| < \delta \leq \varepsilon/2$ forces $|x-2| < \varepsilon/2$

Now for $0 < |x-2| < \delta$ $|x-2| < \varepsilon/2$ and $|x-1| < 2$

so $|f(x)-L| = |x-2||x-1| < (\varepsilon/2) \cdot 2 = \varepsilon$

We have shown that $0 < |x-a| < \delta$ forces $|f(x)-L| < \varepsilon$

so $\lim_{x \to 2} x^2-3x-1 = -3$ is proven. ∎

6. Use either form of the definition of the derivative (your choice) to find the derivative of the function $f(x) = 3 - 2x^2$ at $a = -1$. *(6 points)*

Definition - Derivative

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{or} \quad f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$f(x) = 3-2x^2$ $f(-1) = 1$

$$f'(-1) = \lim_{x \to -1} \frac{f(x) - f(-1)}{x-(-1)}$$

$$= \lim_{x \to -1} \frac{(3-2x^2)-(1)}{x+1}$$

$$= \lim_{x \to -1} \frac{2-2x^2}{x+1}$$

$$= \lim_{x \to -1} \frac{-2(x^2-1)}{x+1}$$

$$= \lim_{x \to -1} \frac{-2(x-1)(x+1)}{x+1}$$

$$= \lim_{x \to -1} -2(x-1)$$

$$\boxed{f'(-1) = 4}$$

$f(-1+h) = 3-2(-1+h)^2 = 3-2(1-2h+h^2)$

$f(-1) = 3-2(-1)^2 = 3-2 = 1$

$$f'(-1) = \lim_{h \to 0} \frac{3-2(1-2h+h^2) - 1}{h}$$

$$= \lim_{h \to 0} \frac{3-2+4h-2h^2-1}{h}$$

$$= \lim_{h \to 0} \frac{4h-2h^2}{h}$$

$$= \lim_{h \to 0} \frac{h(4-2h)}{h}$$

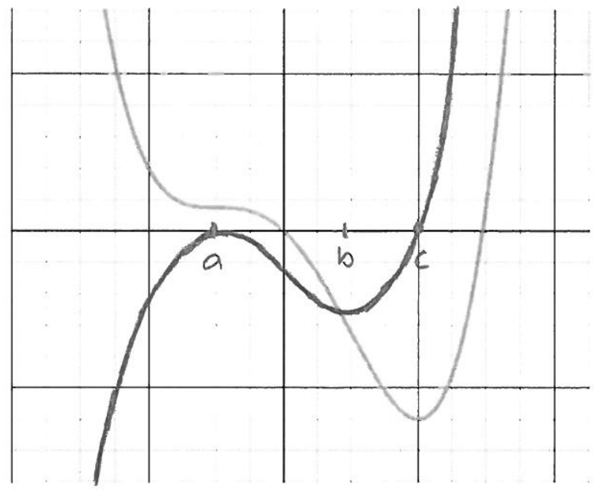
$$= \lim_{h \to 0} 4-2h$$

$$\boxed{f'(-1) = 4}$$

7. The function $f(x)$ is graphed on the coordinate axes below, make a reasonable sketch of the graph of the function $f'(x)$ on the same axes. *(6 points)*

$f'(a), f'(c) = 0$

$f'(b)$ local minimum



8. Differentiate the function $f(x) = 2x + \frac{3}{x} - \frac{4}{x^2}$ and find the equation of the line tangent to $f(x)$ at $x = -1$. *(12 points)*

$f(x) = 2x + 3x^{-1} - 4x^{-2}$

$f'(x) = 2 - 3x^{-2} + 8x^{-3}$

$\boxed{f'(x) = 2 - \frac{3}{x^2} + \frac{8}{x^3}}$

$f(-1) = 2(-1) + \frac{3}{(-1)} - \frac{4}{(-1)^2}$

$= -2 - 3 - 4 = -9$

$f'(-1) = 2 - \frac{3}{(-1)^2} + \frac{8}{(-1)^3}$

$= 2 - 3 - 8 = -9$

tangent line through $(-1, f(-1)) = (-1, -9)$

with slope $f'(-1) = -9$

$y - y_1 = m(x - x_1)$

or $y - f(-1) = f'(-1)(x - (-1))$

$y - (-9) = -9(x+1)$

$y = -9x - 9 - 9$

$\boxed{y = -9x - 18}$

Intended was $P_1(x) = 40x - 0.4x^2 - 240$, it is possible to use $P_2(x) = 40 - 0.4x^2 - 240$
$= -0.4x^2 - 200$

9. A profit is earned when revenue exceeds cost. Suppose the profit function for a snowboard manufacturer is $P(x) = 40 - 0.4x^2 - 240$, where x is the number of snowboards sold. *(12 points)* I will accept solutions using either P_1 or P_2

   (a) Find the exact profit from the sale of the 25^{th} snowboard.

   (b) Find the marginal profit function and use it to estimate the profit from the sale of the 25^{th} snowboard.

(a) $P_1(25) - P_1(24) = (40(25) - 0.4(25)^2 - 240) - (40(24) - 0.4(24)^2 - 240)$
$= 40 - 0.4(625 - 576) = \boxed{\$20.40}$

$P_2(25) - P_2(24) = (-0.4(25)^2 - 200) - (-0.4(24)^2 - 200)$
$= -0.4(625 - 576) = \boxed{-\$19.60}$

(b) $MP_1(x) = P_1'(x) = 40 - 0.8x$ $\quad MP_2(x) = -0.8x$

$MP_1(24) = P_1'(24) = 40 - 0.8(24)$ $\boxed{= \$20.80}$ $\quad MP_2(24) = -0.8(24)$ $\boxed{= -\$19.20}$

10. Calculate the first and second derivatives of the function $f(x) = x\cos x - \sin x$. *(12 points)*

$f'(x) = (x)(-\sin x) + (1)(\cos x) - \cos x$

$\boxed{f'(x) = -x\sin x}$

$f''(x) = (-x)(\cos x) + (-1)(\sin x)$

$\boxed{f''(x) = -x\cos x - \sin x}$

11. Calculate the derivative of the function $y = \tan^2\left(\sqrt{4x-1}\right)$ using the chain rule and other appropriate rules. *(6 points)*

$y = \tan^2(\sqrt{4x-1})$

$y = (\tan(\sqrt{4x-1}))^2$

$\frac{dy}{dx} = 2(\tan(\sqrt{4x-1})) \cdot \sec^2(\sqrt{4x-1}) \cdot \left(\frac{1}{2\sqrt{4x-1}}\right) \cdot 4$

$$\boxed{\frac{dy}{dx} = \frac{4\tan(\sqrt{4x-1})\sec^2(\sqrt{4x-1})}{\sqrt{4x-1}}}$$