Pascal's Triangle and Modular Arithmetic

The array that we call **Pascal's triangle** is a list of rows of natural numbers. The top row is traditionally referred to as row 0 (followed by row 1, row 2, . . .), and the left-hand column is referred to as column 0 (followed by column 1, column 2, . . .).

Row 0, contains a single "1", and each row is one entry longer than the row that precedes it. Each entry is computed by adding the numbers "above" and "above-left". The entry C(5,3) in row 5, column 3 is 10, because:

$$C(5,3) = C(4,2) + C(4,3) = 6 + 4 = 10$$

The "C" in $C(n, r)$ is a mathematical notation for "how many ways can a **combination** of r elements be selected from a set of n elements?" This formula for computing entries is known as Pascal's identity. For example, the first 7 rows of Pascal's triangle are shown in the figure on the right. As you can imagine, this addition quickly gets out of hand! For example, the entry C(25,13) in row 25, column 13 is 5,200,300.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1

Modular addition is a version of addition in which you add two numbers ordinarily and then calculate the remainder of their sum upon division by a given natural number n, called the **modulus**. For example:

$$1 + 1 \equiv 0 \text{ mod } 2 \quad \text{and} \quad 2 + 3 \equiv 1 \text{ mod } 4$$

because 2 leaves a remainder of 0 upon division by 2, and 5 leaves a remainder of 1 upon division by 4. We count the hours on a clock modulo 12; if your watch says 10 o'clock now, then in 7 hours it will be 5 o'clock because:

$$10 + 7 \equiv 5 \text{ mod } 12$$

With the help modular addition, we can continue adding rows to Pascal's triangle without generating unreasonably large numbers. In fact, we can calculate the entries in Pascal's triangle modulo any natural number n we wish, and study the resulting patterns.

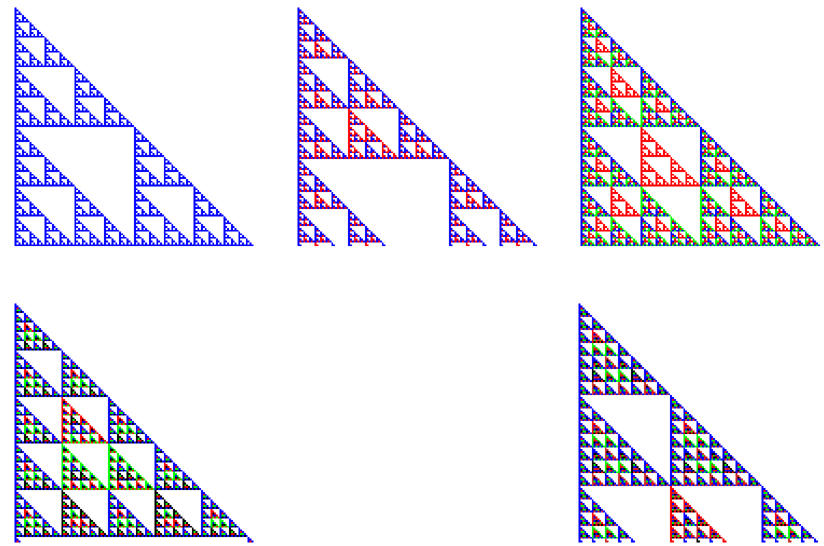
Let's calculate the entries in the first nine rows of Pascal's triangle mod 2, mod 3, mod 4, mod 5, and mod 7:

1	1	1	1	1
1 1	1 1	1 1	1 1	1 1
1 0 1	1 2 1	1 2 1	1 2 1	1 2 1
1 1 1 1	1 0 0 1	1 3 3 1	1 3 3 1	1 3 3 1
1 0 0 0 1	1 1 0 1 1	1 0 2 0 1	1 4 1 4 1	1 4 6 4 1
1 1 0 0 1 1	1 2 1 1 2 1	1 1 2 2 1 1	1 0 0 0 0 1	1 5 3 3 5 1
1 0 1 0 1 0 1	1 0 0 2 0 0 1	1 2 3 0 3 2 1	1 1 0 0 0 1 1	1 6 1 6 1 6 1
1 1 1 1 1 1 1 1	1 1 0 2 2 0 1 1	1 3 1 3 3 1 3 1	1 2 1 0 0 1 2 1	1 0 0 0 0 0 0 1
1 0 0 0 0 0 0 0 1	1 2 1 2 1 2 1 2 1	1 0 0 0 2 0 0 0 1	1 3 3 1 0 1 3 3 1	1 1 0 0 0 0 0 1 1

Do you see any indications of patterns in these triangles? Let's accelerate the process and expose the patterns in a beautiful fashion by

1. Coloring each non-zero number, i.e. 1 = blue, 2 = red, 3 = green, . . .
2. Programming a computer to generate many rows of these colored images.

Below we have generated images of Pascal's triangle mod 2, mod 3, mod 4, mod 5, and mod 7 with 128 rows in each image!



Mathematicians use the patterns in Pascal's triangle in probability, statistics, and general counting problems. Computer scientists and mathematicians use modular arithmetic to allow computers to do arithmetic with extremely large integers, as well as in the construction of highly secure encryption applications.

The Challenge

On a ordinary piece of graphing paper (at least 4 squares per inch), construct 32 rows of Pascal's triangle mod 6, and color the entries 1, 2, 3, 4, and 5 with 5 different colors. Leave the 0 entries uncolored. What is the connection between the mod 2, mod 3, and mod 6 triangles? Write your name clearly on the paper and submit your coloring when you check in on the day of the competition.

Further information about fractals, including links to many fractal web sites can be found at *http://www.delta.edu/˜bdredman*. Color versions of this handout are available at *http://www.delta.edu/˜math/projects/competition/competition.html*.